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ELEMENTARY TREATISE

ON

DETERMINANTS.

BY

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PREFACE.

THIS book was originally prepared for the use of a class of students about to enter upon a course of modern Co-ordinate Geometry. The favor with which it was received has led to its publication.

Recent advances in mathematics demand an acquaintance with the elementary principles of Determinants. In the hope that this little treatise will meet the growing want, it is submitted to the teachers of mathematics in our colleges and schools of science, as well as to all students of the modern methods of mathematical investigation.

The author takes this occasion to express his grateful acknowledgment of the service rendered him by Messrs. Fiske, Stabler, and Cromwell, Fellows in Science in this college. They have made many valuable suggestions, and have aided in reading the proof-sheets as they came from the press.

COLUMBIA COLLEGE,
Feb. 22, 1887.

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DETERMINANTS.

Definitions and Principles.

1. A **function** of two or more quantities is an expression whose value depends upon those quantities: thus, $ad - bc$ is a function of a, b, c, and d.

The expression just given may also be written

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

in which form it is called a **determinant.**

We have then the equation

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc,$$

in which the *first* member is a determinant and the *second* member is its **development,** or, its expression, in ordinary algebraic notation.

The method of finding the development in this particular case is obvious.

EXAMPLES.

1. $\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 3 \times 5 - 4 \times 2 = 7.$

2. $\begin{vmatrix} -2 & 3 \\ 6 & 10 \end{vmatrix} = -2 \times 10 - 6 \times 3 = -38.$

3. $\begin{vmatrix} 4 & -2 \\ 3 & 5 \end{vmatrix} = 4 \times 5 - 3 \times (-2) = 26.$

4. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$

5. $\begin{vmatrix} 5 & 7 \\ 1 & 4 \end{vmatrix} = 13; \quad \begin{vmatrix} -5 & 7 \\ 1 & -4 \end{vmatrix} = 13.$

The quantities on which a determinant depends are called **elements**: thus, in Example 2, the elements are -2, 3, 6, and 10.

The elements in any horizontal line constitute a **row**; those in any vertical line make up a **column.** In every determinant there are as many rows as columns; hence, the number of elements in a determinant is a *square number.*

Determinants are divided into **orders** according to the number of rows they contain: thus, the determinants already mentioned are of the *second order.*

It is found convenient to denote the elements of determinants by letters with subscript numbers as in Example 4. In what follows we shall, in general, write them so that the letters in each column shall be the same, the rows being denoted by the subscript numbers.

Determinants of the Third Order.

2. The ordinary form of a determinant of the third order is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

and its *development* is

$$x_1y_2z_3+x_2y_3z_1+x_3y_1z_2-x_3y_2z_1-x_2y_1z_3-x_1y_3z_2.$$

If we examine the 6 terms of the above development, we see that each term contains one and only one element from each column, and one and only one element from each row. We also see that one half of the terms are positive and the other half negative.

Before deducing rules for the development of a determinant of any order, we shall explain the *practical* rule for finding the development of a determinant of the *third* order.

The *first* positive term of the development is the product of the elements of the **downward,** or **principal diagonal,** x_1,y_2,z_3; the *second* positive term is the product of the elements in the *broken* parallel line x_2,y_3, supplemented by z_1; the *third* positive term is the product of the elements in the *broken* parallel line x_3, supplemented by the elements y_1,z_2. The *first* negative term is found by using the elements of the **upward,** or **secondary diagonal,** x_3,y_2,z_1; the *second* negative term by using the *broken* parallel line x_2, y_1, supplemented by the element z_3; the *third* negative term by using the *broken* parallel x_1, supplemented by the elements y_3, z_2.

EXAMPLES

1. $$\begin{vmatrix} 2 & 3 & 1 \\ 1 & 7 & 4 \\ 2 & 2 & 3 \end{vmatrix} = 42+2+24-14-9-16 = 29.$$

2. $$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{vmatrix} = 0; \quad \begin{vmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 6.$$

3. $$-2\begin{vmatrix} 3 & 0 & 1 \\ -5 & 6 & 2 \\ 4 & 2 & 0 \end{vmatrix} = -2(-10-24-12) = 92.$$

4. $$3\begin{vmatrix} 4 & -5 & 6 \\ -2 & 4 & 0 \\ 1 & -2 & 3 \end{vmatrix} = 3(48+24-24-30) = 54.$$

5. $$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}^{=0} -4\begin{vmatrix} 4 & 5 & 2 \\ -1 & 2 & -3 \\ 6 & -4 & 5 \end{vmatrix} = 356.$$

6. $$\begin{vmatrix} 1 & -1 & 1 \\ 4 & -3 & 0 \\ 3 & 2 & -5 \end{vmatrix}^{=?} -3\begin{vmatrix} 4 & -1 & -2 \\ 0 & 3 & 0 \\ 3 & -7 & 4 \end{vmatrix} = -186.$$?

7. $$\begin{vmatrix} 2c & a+b+c & a+b+c \\ a+b+c & 2a & a+b+c \\ a+b+c & a+b+c & 2b \end{vmatrix} = 8abc.$$

Solve the following equations:

8. $$\begin{vmatrix} x & -4 & 1 \\ -6 & 3 & -2 \\ x & 2 & 1 \end{vmatrix} = 0. \quad \textit{Ans. } x = 3.$$

9. $$\begin{vmatrix} 1 & 1 & 1 \\ a & x & c \\ b & b & x \end{vmatrix} = 0. \quad \textit{Ans. } x = a,\ x = b.$$

10. $$\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} + \begin{vmatrix} b & b & x \\ b & x & b \\ x & b & b \end{vmatrix} = 0.$$

$$Ans.\ x = \frac{2}{3} \cdot \frac{a^2 + ab + b^2}{a + b}.$$

11. Write, $ahm - fhk + clk - cbm + fbd - ald$, in the form of a determinant.

$$Ans. \begin{vmatrix} a & b & k \\ c & h & d \\ f & l & m \end{vmatrix}.$$

Note.—There are other methods of writing determinants than that already given, when the elements are expressed by letters with subscript numbers. Thus, the determinant at the head of this article may be written

$$| x_1 y_2 z_3 |.$$

This represents the typical term formed by taking the product of the elements of the principal diagonal; we shall see hereafter that all the other terms of the development can be got from this one, by a suitable change in the order of the subscript numbers, with a due regard to signs. The same determinant is sometimes written

$$\Sigma \pm (x_1 y_2 z_3).$$

Determinants of any Order.

3. A determinant having n rows and n columns is called a determinant of the n^{th} order, n being any positive whole number.

It is *agreed* that every such expression shall stand for the algebraic sum of all the products that can be formed by taking one and only one element from each row and one and only one element from each column, the sign of each product being determined in the following manner. The elements of any product being written so that the *letters* shall stand in the order of the columns from which they are taken, the numbers indicating the rows from which they are taken are written in a line; then the **sign factor** of the product is $(-1)^p$, p being the number of **inversions** of order in the line.

To explain what is meant by an **inversion**, take the line of numbers 3, 2, 1. In this case 3 occupies an inverted position with respect to 2, and also with respect to 1; 2 is inverted with respect to 1: hence, there are three inversions, that is, $p = 3$.

In the line 4, 3, 1, 2 there are five inversions.

EXAMPLES.

In the determinant of the 4th order

$$| a_1 b_2 c_3 d_4 |,$$

1. What is the sign factor of the term $a_4 b_1 c_3 d_2$?

Solution.—Here $p = 4$, hence the sign factor is $(-1)^4 = +1$.

2. What is the sign of $a_3 b_2 c_4 d_1$? *Ans.* $+$.

3. What is the sign of $a_4 b_3 c_1 d_2$? *Ans.* $-$.

The principles stated in this article, which are purely conventional, afford a full and complete explanation of the meaning of the word **determinant**.

The Number of Terms.

4. Let us consider a determinant of the n^{th} order written as explained in Art. 1, the basal letters being a, b, c, d, etc. Then, from Art. 3, every product will contain all the basal letters, and their products will differ from each other only in the positions of the subscript numbers. In the aggregate of products the subscript numbers must occur in every possible order; hence, the whole number of terms must be equal to the whole number of permutations of the subscript numbers. A determinant of the 2d order then has $1 \cdot 2$, or 2 terms; one of the 3d order has $1 \cdot 2 \cdot 3$, or 6 terms; one of the 4th order has $1 \cdot 2 \cdot 3 \cdot 4$, or 24 terms; and, in general, one of the n^{th} order has $1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$ terms.

Signs of the Terms.

5. As an illustration, let us take a determinant of the 4th order.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}; \text{ or } |a_1 b_2 c_3 d_4|.$$

To determine the sign of a term, say $a_3 b_2 c_4 d_1$. Here the elements stand in the order of the columns from which they are taken and the subscript numbers in the order of the corresponding rows. Now the line 3 2 4 1 has 4 inversions, hence the term is positive. In like manner, the term $a_3 b_4 c_2 d_1$ is *negative.*

If we interchange the subscript numbers of any two adjacent letters of any product, we change the sign of the product; thus, if we change the places of 4 and 2 in the last example, we have $a_3 b_2 c_4 d_1$, which is positive. Now both $a_3 b_4 c_2 d_1$ and $a_3 b_2 c_4 d_1$ are terms of the development of the given determinant; hence, for every positive term there is a negative term; that is, there are always as many positive terms as there are negative ones.

We have taken, as an illustration, a determinant of the 4th order, but the conclusions would not be changed if we were to take a determinant of any other order; hence, in any determinant there are as many positive terms as there are negative ones.

Properties of Determinants.

6. *First.* If the first row of a determinant is made the first column of a second determinant, the second row of the first, the second column of the second, and so on till all the rows of the first are made columns of the second, then will the two determinants be equal. This is obvious from the definition, Art. 3. Thus, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

a result that may be verified by making the development.

7. *Secondly.* Multiplying all the elements of any row (or column) by any number is equivalent to multiplying the determinant by that number. Thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ ra_2 & rb_2 & rc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = r \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

or, $$| a_1 \; rb_2 \; c_3 | = r \, | a_1 \; b_2 \; c_3 |.$$

For in the development of the first member every term must contain one element and only one taken from the second row; that is, it must contain the factor r: in like manner it may be shown that the theorem is true in all cases.

Note.—The proposition is true whether r is positive or negative, entire or fractional.

EXAMPLE.

$$\begin{vmatrix} a_1 & b_1 & mc_1 \\ a_2 & b_2 & mc_2 \\ a_3 & b_3 & mc_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{1}{m} \cdot \begin{vmatrix} a_1 & mb_1 & c_1 \\ ma_2 & m^2b_2 & mc_2 \\ a_3 & mb_3 & c_3 \end{vmatrix}.$$

8. *Thirdly.* Interchanging any two consecutive rows of a determinant is equivalent to changing the sign of the determinant from + to —, or from — to +. For, every term contains an element from each of the two rows, and inasmuch as the interchange is equivalent to an exchange of two adjacent subscript numbers in each term, it follows that every term, and consequently the whole determinant, is changed from + to —, or from — to +.

Because rows may be taken as columns and columns as rows, it follows that the interchange of two adjacent columns is equivalent to changing the sign of the determinant from + to —, or from — to +.

EXAMPLE.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix} = - \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix};$$

or, $$|a_1b_2c_3| = -|a_1b_3c_2| = -|a_1c_2b_3|.$$

9. *Fourthly.* Interchanging any two rows of a determinant is equivalent to changing the sign of the determinant.

For, suppose there are m intervening rows. Then beginning with the lower row, we can by m consecutive changes, and as many changes of sign, bring it to the place next below the upper row; now beginning with upper row and moving downward by $m+1$ successive steps, and as many changes of sign, we can bring it to the place originally held by the lower row, that is, the two rows will have changed places, and the determinant will have changed sign $2m+1$ times, which is the same thing as changing sign *once.* Interchanging any two columns is equivalent to changing the sign of the determinant.

10. *Fifthly.* Any element of a determinant may be brought to the upper left-hand corner as follows:

Suppose the element to be in the 4th row and 3d column. By 3 interchanges and 3 changes of sign the 4th row may be made the 1st, and then by two corresponding changes the 3d column may be made the 1st. In this case the determinant changes sign $3+2$, or 5 times, that is, its sign factor becomes $(-1)^5$.

EXAMPLE.

To bring c_4 to the upper left-hand corner:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = (-1)^5 \begin{vmatrix} c_4 & a_4 & b_4 & d_4 \\ c_1 & a_1 & b_1 & d_1 \\ c_2 & a_2 & b_2 & d_2 \\ c_3 & a_3 & b_3 & d_3 \end{vmatrix};$$

or, $$|a_1b_2c_3d_4| = -|c_4a_1b_2d_3|.$$

11. *Sixthly.* If two columns (or two rows) of a determinant are identical, the determinant is equal to 0. For, from Art. 9, an interchange of these columns (or rows) will change the sign of the determinant from + to —, or from — to +. But it is obvious that such an interchange will not change the *value* of the determinant. Now, 0 is the only quantity whose sign can be changed from + to —, or from — to +, without change of value; hence the given determinant is equal to 0. Thus,

$$\begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} = 0; \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0,$$

which may be verified by development.

Cor. If the elements of any row, or column, are equimultiples of the corresponding elements of any other row, or column, the determinant is 0. Thus,

$$\begin{vmatrix} 3 & 2 & 4 \\ 6 & 4 & 8 \\ 1 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 & 4 \\ 3 & 2 & 4 \\ 1 & 2 & 3 \end{vmatrix} = 0; \text{ and } \begin{vmatrix} 4 & 8 & 1 \\ 1 & 2 & 5 \\ 2 & 4 & 6 \end{vmatrix} = 0.$$

Minors—Cofactors—Reduction.

12. If we *strike out,* or **delete** any number of rows and the same number of columns of a determinant, the remaining elements taken in order constitute a determinant which is called a **minor**; the elements common to the deleted rows and columns taken in order constitute a determinant which is also called a **minor**: these minors are said to be **complementary**.

If, for example, we delete the first two rows and the first two columns of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

the *complementary minors* will be

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \text{ and } \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}.$$

In forming the product of two complementary minors, the sign factor of the result must be taken into account; when this is done the complementary minors are called **Cofactors** of the result; thus, when we delete the first line and the first column of (1), we have the cofactors

$$a_1 \text{ and } \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}.$$

It is readily seen that the cofactor of a_1 does not contain any element of the first row or of the first column. It must then be made up of the different products that can be formed from its elements in accordance with Art. 3. If, then, each of these products be multiplied by a_1, the resulting products will give all the terms of the original determinant that contain a_1.

Now, let us denote the cofactor of any element by the corresponding *capital letter*, with the *same subscript number;* we shall then have for the aggregate of all the terms of the original determinant that contain a_1 the expression a_1A_1.

If we now transfer the second row of (1) to the first (as explained in Art 8), thus making the determinant negative, we find in like manner that the aggregate of all the terms containing a_2 is $-a_2A_2$.

Transferring the third row to the top, we find the aggregate of all the terms containing a_3 to be a_3A_3, and in like manner we find the aggregate of all the terms containing a_4 to be $-a_4A_4$.

Now the given determinant comprises all the terms containing the a elements, and no other; hence, replacing cofactors by their values, and using proper signs, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}$$

$$+ a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

and similarly for determinants of any order. Hence, for reducing a determinant to a lower order, we have the

RULE.

Multiply each element of the first column by its complementary minor, making the products alternately plus and minus; then take the algebraic sum of the results.

Any *column*, or any *row*, may be used in making the reduction if we apply the proper signs to the partial products.

The sign factor of any partial product is, from what precedes, equal to $(-1)^{m+n}$, in which m denotes the row in which the element is found and n the column: thus, the sign factor of d_3D_3 is $(-1)^7$, or -1, because d_3 is in the 3d row and 4th column.

Cor. If all the elements of a row, or column, except one are 0, the value of the determinant is equal to the product of that element by its cofactor, the proper sign factor being used: thus

$$\begin{vmatrix} 1 & 2 & 2 & 5 \\ 2 & 1 & 3 & 6 \\ 0 & 0 & 7 & 0 \\ 4 & 2 & 2 & 4 \end{vmatrix} = (-1)^6 \times 7 \begin{vmatrix} 1 & 2 & 5 \\ 2 & 1 & 6 \\ 4 & 2 & 4 \end{vmatrix} = 168.$$

EXAMPLES.

1. $$\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 3 & 2 & 5 \end{vmatrix} = 1 \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} + 3 \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = 4.$$

2. $$\begin{vmatrix} 6 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 2 & 4 \end{vmatrix} = 6 \begin{vmatrix} 4 & 5 \\ 2 & 4 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 26.$$

3. $$\begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & -3 \\ -4 & 2 & 5 \end{vmatrix} = 3 \begin{vmatrix} 1 & -3 \\ 2 & 5 \end{vmatrix} - 2 \begin{vmatrix} -1 & 2 \\ 2 & 5 \end{vmatrix} - 4 \begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} = 47.$$

4. $$\begin{vmatrix} 15 & 17 & 16 \\ 12 & 18 & 14 \\ 19 & 17 & 13 \end{vmatrix} = 15\begin{vmatrix} 18 & 14 \\ 17 & 13 \end{vmatrix} - 12\begin{vmatrix} 17 & 16 \\ 17 & 13 \end{vmatrix} + 19\begin{vmatrix} 17 & 16 \\ 18 & 14 \end{vmatrix}$$
$$= -398.$$

5. $$\begin{vmatrix} 10 & 4 & 17 & 13 \\ 4 & 2 & 8 & 6 \\ 3 & -1 & 8 & 1 \\ 7 & 5 & 20 & 17 \end{vmatrix} = 10\begin{vmatrix} 2 & 8 & 6 \\ -1 & 8 & 1 \\ 5 & 20 & 17 \end{vmatrix} - 4\begin{vmatrix} 4 & 17 & 13 \\ -1 & 8 & 1 \\ 5 & 20 & 17 \end{vmatrix}$$
$$+3\begin{vmatrix} 4 & 17 & 13 \\ 2 & 8 & 6 \\ 5 & 20 & 17 \end{vmatrix} - 7\begin{vmatrix} 4 & 17 & 13 \\ 2 & 8 & 6 \\ -1 & 8 & 1 \end{vmatrix} = 124.$$

6. $$\begin{vmatrix} 3 & 4 & 1 & 2 \\ 6 & 9 & 7 & 5 \\ 7 & 10 & 5 & 8 \\ 4 & 2 & 9 & 3 \end{vmatrix} + \begin{vmatrix} 7 & 3 & 9 & 4 \\ 3 & 9 & 4 & 7 \\ 2 & 6 & 1 & 5 \\ 10 & 7 & 14 & 10 \end{vmatrix} = -440.$$

7. Because

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

we have conversely

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

8. $$2\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 1 & 5 \\ 5 & 6 \end{vmatrix} + 4\begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 5 \\ 3 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}.$$

The principles of this article enable us to reduce determinants of a higher order to others of a lower order, and the reverse.

Application to the solution of Simultaneous Equations of the First Degree.

13. Let us begin with two simultaneous equations,

$$a_1x+b_1y=m_1, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$a_2x+b_2y=m_2. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By the ordinary methods of algebra, we have

$$(a_1b_2-a_2b_1)x=(m_1b_2-m_2b_1),$$ = [illegible]

which may be written

$$x=\frac{\begin{vmatrix} m_1 & b_1 \\ m_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

and because the given equations are symmetrical we at once write

$$y=\frac{\begin{vmatrix} a_1 & m_1 \\ a_2 & m_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Here we see that *either unknown is equal to a fraction whose denominator is the determinant formed by taking the coefficients of the unknowns in order, and whose numerator may be obtained from the denominator by replacing the coefficients of the required unknown by the corresponding second members.*

EXAMPLES.

1. $\left.\begin{aligned} 2x+3y &= 13 \\ x+2y &= 8 \end{aligned}\right\}.$

$$x = \frac{\begin{vmatrix} 13 & 3 \\ 8 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}} = 2; \quad y = \frac{\begin{vmatrix} 2 & 13 \\ 1 & 8 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}} = 3.$$

2. $\left.\begin{aligned} 3x+\ y &= 3 \\ 4x+2y &= 5 \end{aligned}\right\}.$ *Ans.* $x = \frac{1}{2},\ y = \frac{3}{2}.$

Find the value of x in each of the following groups:

3. $\left.\begin{aligned} 2x+4y &= 12 \\ 3x-2y &= 10 \end{aligned}\right\}.$ *Ans.* $x = \frac{\begin{vmatrix} 12 & 4 \\ 10 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 3 & -2 \end{vmatrix}} = 4.$

y = 1

4. $\left.\begin{aligned} 3x-2y &= 5 \\ -x-y &= -5 \end{aligned}\right\}.$ *Ans.* $x = \frac{\begin{vmatrix} 5 & -2 \\ -5 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -1 & -1 \end{vmatrix}} = 3.$

y = 2

Find the value of y in the following group:

5. $\left.\begin{aligned} 5x-2y &= 6 \\ 3x+4y &= 14 \end{aligned}\right\}$ *Ans.* $y = 2.$

x = 2

14. Let us next take three simultaneous equations:

$$a_1x+b_1y+c_1z = m_1 \quad \ldots\ldots \quad (1)$$
$$a_2x+b_2y+c_2z = m_2 \quad \ldots\ldots \quad (2)$$
$$a_3x+b_3y+c_3z = m_3 \quad \ldots\ldots \quad (3)$$

multiply

$$(1) \text{ by } \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}; \quad (2) \text{ by } -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}; \quad (3) \text{ by } \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix};$$

and add the resulting equations member by member.

The resulting coefficient of x will be

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The resulting coefficient of y will be

$$b_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix},$$

which, from Art. 11, is equal to 0. In the same manner, it may be shown that the resulting coefficient of z is equal to 0.

The resulting value of the second member is

$$m_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - m_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + m_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}.$$

Hence, we have

$$x = \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{|m_1 b_2 c_3|}{|a_1 b_2 c_3|}.$$

The values of y and z may be obtained in the same way, or they may be written out symmetrically from the value of x.

Ience, we see that the rule given in Art. 13 holds l for the solution of three simultaneous equations.

If we have four simultaneous equations,

$$a_1x+b_1y+c_1z+d_1w=m_1 \quad . \; . \; . \quad (1)$$
$$a_2x+b_2y+c_2z+d_2w=m_2 \quad . \; . \; . \quad (2)$$
$$a_3x+b_3y+c_3z+d_3w=m_3 \quad . \; . \; . \quad (3)$$
$$a_4x+b_4y+c_4z+d_4w=m_4 \quad . \; . \; . \quad (4)$$

we multiply equations (1), (2), (3), and (4) in order by

$$\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}, \; -\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}, \; \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix}, \text{ and } -\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix};$$

adding the results and proceeding as before, we find that the rule given in Art. 13 holds true. By an analogous course of reasoning it may be shown that the rule holds true for any number of equations of the first degree.

EXAMPLES.

Find the value of x in each of the following groups:

1. $\left.\begin{aligned} 3x+4y-2z &= 10 \\ 5x-2y+3z &= 16 \\ 4x+2y+2z &= 22 \end{aligned}\right\}.$

$$\textit{Ans. } x = \begin{vmatrix} 10 & 4 & -2 \\ 16 & -2 & 3 \\ 22 & 2 & 2 \end{vmatrix} \div \begin{vmatrix} 3 & 4 & -2 \\ 5 & -2 & 3 \\ 4 & 2 & 2 \end{vmatrix} = 2.$$

2. $\left.\begin{aligned} x+\ y+\ z &= 6 \\ 5x+2y-3z &= 0 \\ 2x+\ y-\ z &= 1 \end{aligned}\right\}.$

$$\textit{Ans. } x = \begin{vmatrix} 6 & 1 & 1 \\ 0 & 2 & -3 \\ 1 & 1 & -1 \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ 5 & 2 & -3 \\ 2 & 1 & -1 \end{vmatrix} = 1.$$

3. $$\left.\begin{aligned} 2x-2y+3z &= 16 \\ 3x+5y-2z &= 6 \\ 4x+3y-4z &= -1 \end{aligned}\right\} \cdot \qquad \textit{Ans. } x = 3.$$

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7

Consistence of Equations—Eliminants.

15. In the groups of equations treated of in the last article, the number of independent equations equals the number of unknowns. In every such group the equations are said to be **consistent**, or sometimes they are said to **consist.**

When the number of equations in a group is greater by 1 than the number of unknowns, the equations cannot *consist* unless the relation between the coefficients of the unknowns is such as to make one of the group depend upon the others. The expression of this relation, which is called the **eliminant of the group,** may be found by combining the equations of the group so as to eliminate all of the unknowns.

Let us consider the group

$$a_1x+b_1y+m_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$a_2x+b_2y+m_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$a_3x+b_3y+m_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which the absolute terms are written in the first members.

From equations (1) and (2), we have, by the last article,

$$x = \frac{\begin{vmatrix} -m_1 & b_1 \\ -m_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} b_1 & m_1 \\ b_2 & m_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

and
$$y = \frac{\begin{vmatrix} a_1 & -m_1 \\ a_2 & -m_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = -\frac{\begin{vmatrix} a_1 & m_1 \\ a_2 & m_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Substituting these values of x and y in equation (3) and clearing of fractions, we have

$$a_3 \begin{vmatrix} b_1 & m_1 \\ b_2 & m_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & m_1 \\ a_2 & m_2 \end{vmatrix} + m_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0;$$

or,
$$\begin{vmatrix} a_1 & b_1 & m_1 \\ a_2 & b_2 & m_2 \\ a_3 & b_3 & m_3 \end{vmatrix} = 0, \quad . \ . \ . \ . \ . \ . \quad (4)$$

which is the required *eliminant.*

The first member of (4), the determinant formed by taking the coefficients and the absolutes in order, is called the **determinant of the group.**

Hence, *the eliminant is formed by writing the determinant of the group equal to* 0.

By a similar course of reasoning, it may be shown that the above rule holds good when there are n equations and $n-1$ unknowns, n being any positive whole number.

Homogeneous Equations of the First Degree.

16. Homogeneous equations of the first degree are those in which every term contains an unknown multiplied by a known factor: thus, $a_1x+b_1y+c_1z = 0$ is *homogeneous,* but $a_1x+b_1y+m_1 = 0$ is *not homogeneous.*

When the number of homogeneous equations of a group is equal to the number of unknowns, the values of the unknowns are 0, and the equations vanish unless the relation between the coefficients is such as to make one of the equations depend upon the others. In this case, the equations will, in general, enable us to find the values of the ratios of the unknowns, that is, to find the values of all the unknowns except one in terms of that one to which we may assign any value at pleasure.

Let us consider the group

$$\left.\begin{array}{l} a_1x+b_1y+c_1z=0 \\ a_2x+b_2y+c_2z=0 \\ a_3x+b_3y+c_3z=0 \end{array}\right\} \quad . \; . \; . \; . \quad (1)$$

From this we have at once, on the supposition that z is not equal to 0,

$$\left.\begin{array}{l} a_1\dfrac{x}{z}+b_1\dfrac{y}{z}+c_1=0 \\ a_2\dfrac{x}{z}+b_2\dfrac{y}{z}+c_2=0 \\ a_3\dfrac{x}{z}+b_3\dfrac{y}{z}+c_3=0 \end{array}\right\} \quad . \; . \; . \; . \; . \quad (2)$$

and these will *consist* when

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad . \; . \; . \; . \; . \quad (3)$$

Hence, equation (3) expresses the condition that one equation of group (1) shall depend upon the others. It is the eliminant of group (2), and consequently of group (1).

The principle shown above may be extended to groups of homogeneous equations containing n equations and n unknowns, n being *any* positive whole number.

17. When the number of homogeneous equations of a group is one less than the number of unknowns, we can, as already stated, find the values of the *ratios* of the unknowns.

Let us take the group

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= 0 \\ a_2x+b_2y+c_2z &= 0 \end{aligned}\right\} \quad . \; . \; . \; . \quad (1)$$

from which we have at once

$$a_1\frac{x}{z} + b_1\frac{y}{z} = -c_1 \quad . \; . \; . \; . \; . \quad (2)$$

$$a_2\frac{x}{z} + b_2\frac{y}{z} = -c_2 \quad . \; . \; . \; . \; . \quad (3)$$

Solving (2) and (3), by the method of Art. 13, we have

$$\frac{x}{z} = \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } \frac{y}{z} = \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \therefore \frac{x}{y} = \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}};$$

or, $$x : y : z :: \begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix} : \begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

which may be written

$$x : y : z :: \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \; . \; . \; . \quad (4)$$

EXAMPLES.

Find the ratios of the unknowns in the following groups:

1. $2x-3y+4z=0,$

$$x+2y-5z=0.$$

Comparing with group (1) above, we see that

$a_1=2,\ b_1=-3,\ c_1=4,\ a_2=1,\ b_2=2,$ and $c_2=-5$;

hence, from formula (4), we have

$$Ans.\quad x:y:z::\begin{vmatrix}-3 & 4\\ 2 & -5\end{vmatrix}:-\begin{vmatrix}2 & 4\\ 1 & -5\end{vmatrix}:\begin{vmatrix}2 & -3\\ 1 & 2\end{vmatrix},$$

or, $\quad x:y:z::7:14:7::1:2:1.$

Note.—If we make $z=2$, we have $x=2$ and $y=4$: if $z=5$, we have $x=5$ and $y=10$: and so on.

2. $x+y+z=0,$

$$ax+by+cz=0,$$

Ans.

$$x:y:z::\begin{vmatrix}1 & 1\\ b & c\end{vmatrix}:-\begin{vmatrix}1 & 1\\ a & c\end{vmatrix}:\begin{vmatrix}1 & 1\\ a & b\end{vmatrix}::c-b:a-c:b-a.$$

3. $3x-4y-2z=0,$

$$7x-9y-7z=0.$$

Ans. $x:y:z::10:7:1.$

If we have the group

$$\begin{aligned}a_1x+b_1y+c_1z+d_1u&=0,\\ a_2x+b_2y+c_2z+d_2u&=0,\\ a_3x+b_3y+c_3z+d_3u&=0,\end{aligned}$$

we may deduce as before,

$$x:y:z:u::|b_1c_2d_3|:-|a_1c_2d_3|:|a_1b_2d_3|:-|a_1b_2c_3|.$$

4. $$\begin{aligned} 2x+3y-4z-10u &= 0, \\ 3x-4y+2z-\ 5u &= 0, \\ 4x-2y+3z-21u &= 0. \end{aligned}$$

Ans. $x : y : z : u ::$

$$\begin{vmatrix} 3 & -4 & -10 \\ -4 & 2 & -5 \\ -2 & 3 & -21 \end{vmatrix} : -\begin{vmatrix} 2 & -4 & -10 \\ 3 & 2 & -5 \\ 4 & 3 & -21 \end{vmatrix} : \begin{vmatrix} 2 & 3 & -10 \\ 3 & -4 & -5 \\ 4 & -2 & -21 \end{vmatrix} : -\begin{vmatrix} 2 & 3 & -4 \\ 3 & -4 & 2 \\ 4 & -2 & 3 \end{vmatrix},$$

or, $x : y : z : u :: 295 : 236 : 177 : 59 :: 5 : 4 : 3 : 1.$

Note.—If we make $u = 2$, we have $x = 10$, $y = 8$, and $z = 6$; for $u = 3$, we have $x = 15$, $y = 12$, $z = 9$; and so on. There are an infinite number of sets of values that will satisfy the equations.

Many indirect examples may be made direct by simple devices.

5. Find the ratios of a, b, and c from the relation

$$\frac{x}{mb+nc-la} = \frac{y}{nc+la-mb} = \frac{z}{la+mb-nc}.$$

Solution.—Introduce a new unknown by assuming that each ratio is equal to $-\frac{1}{u}$; we then have, after arranging with respect to a, b, and c,

$$\begin{aligned} -la+mb+nc+xu &= 0, \\ la-mb+nc+yu &= 0, \\ la+mb-nc+zu &= 0. \end{aligned}$$

Then, from the formula, we have

$a : b : c : u ::$

$$\begin{vmatrix} m & n & x \\ -m & n & y \\ m & -n & z \end{vmatrix} : -\begin{vmatrix} -l & n & x \\ l & n & y \\ l & -n & z \end{vmatrix} : \begin{vmatrix} -l & m & x \\ l & -m & y \\ l & m & z \end{vmatrix} : -\begin{vmatrix} -l & m & n \\ l & -m & n \\ l & m & -n \end{vmatrix}.$$

Neglecting the fourth term of each member and developing the first three terms of the second member, we have, after dividing by 2,

$$a : b : c :: mn(z+y) : ln(x+z) : lm(x+y).$$

6. What condition will render the following continued equation consistent?

$$a_1x+b_1y+c_1z = a_2x+b_2y+c_2z = a_3x+b_3y+c_3z$$
$$= a_4x+b_4y+c_4z.$$

HINT.—Put each trinomial equal to $-u$, whence

$$\begin{aligned} a_1x+b_1y+c_1z+1\cdot u &= 0, \\ a_2x+b_2y+c_2z+1\cdot u &= 0, \\ a_3x+b_3y+c_3z+1\cdot u &= 0, \\ a_4x+b_4y+c_4z+1\cdot u &= 0. \end{aligned}$$

Now, the eliminant of this group is

$$\begin{vmatrix} a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ a_3 & b_3 & c_3 & 1 \\ a_4 & b_4 & c_4 & 1 \end{vmatrix} = 0,$$

and this is the required condition.

Additional Properties of Determinants.

18. If all the elements of any column (or row) of a determinant are binomials, the determinant may be resolved into two partial determinants, *one* of which is derived from the given determinant by dropping all the *second* terms of the binomial elements, and the other by dropping all the *first* terms.

For, from the definition, each term in the development of the determinant will contain one and only one of these binomials as factors, and is therefore capable of being resolved into two monomial terms, *one* corresponding to the first term of the binomial and *the other* to the second term. The aggregate of those corresponding to the first terms of the binomials will constitute a determinant which is independent of the second terms of the binomials, and the remaining terms will constitute a determinant which is independent of the first terms of the binomials.

Illustration.—Let us consider the determinant

$$\begin{vmatrix} a_1 & m_1+n_1 & c_1 \\ a_2 & m_2+n_2 & c_2 \\ a_3 & m_3+n_3 & c_3 \end{vmatrix}.$$

Developing this by the method of Art. 2, and then resolving each term into parts, we have

$$\begin{aligned} +a_1(m_2+n_2)c_3 &= +a_1m_2c_3+a_1n_2c_3, \\ +a_2(m_3+n_3)c_1 &= +a_2m_3c_1+a_2n_3c_1, \\ +a_3(m_1+n_1)c_2 &= +a_3m_1c_2+a_3n_1c_2, \\ -a_3(m_2+n_2)c_1 &= -a_3m_2c_1-a_3n_2c_1, \\ -a_2(m_1+n_1)c_3 &= -a_2m_1c_3-a_2n_1c_3, \\ -a_1(m_3+n_3)c_2 &= -a_1m_3c_2-a_1n_3c_2. \end{aligned}$$

Summing these results by columns and using the determinant notation, we have

$$\begin{vmatrix} a_1 & m_1+n_1 & c_1 \\ a_2 & m_2+n_2 & c_2 \\ a_3 & m_3+n_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & m_1 & c_1 \\ a_2 & m_2 & c_2 \\ a_3 & m_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & n_1 & c_1 \\ a_2 & n_2 & c_2 \\ a_3 & n_3 & c_3 \end{vmatrix},$$

a result which conforms to the enunciation at the head of this article.

EXAMPLES.

1. $$\begin{vmatrix} 1 & 3+2 & 2 \\ 4 & 2+5 & 1 \\ 3 & 5+7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 4 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 2 \\ 4 & 5 & 1 \\ 3 & 7 & 4 \end{vmatrix}.$$

2. $$\begin{vmatrix} 1 & 2 & 3 \\ 3+1 & 2+2 & 3+4 \\ 5 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \\ 5 & 3 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 5 & 3 & 6 \end{vmatrix}$$

3. $$\begin{vmatrix} 2 & 5 & 3 \\ 3 & 7 & 2 \\ 4 & 9 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 3 \\ 3 & 2 & 2 \\ 4 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 3 \\ 3 & 5 & 2 \\ 4 & 7 & 1 \end{vmatrix}.$$

NOTE.—Here we have resolved the second column into parts, making $5 = 2+3$, $7 = 2+5$, and $9 = 2+7$.

4. $$\begin{vmatrix} 2 & 3 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 1 \\ 2 & 6 & 3 \\ 3 & 9 & 1 \end{vmatrix}.$$

NOTE.—Here the first two determinants have their first and third columns identical, and we have added the corresponding elements in their second columns.

Extension of the Preceding Principle.

19. The principle of Art. 18 is capable of extension to the case in which the elements of one or more columns are trinomials, quadrinomials, etc. Thus,

1. $$\begin{vmatrix} 3 & 2+3+4 & 2 \\ 2 & 1+3+5 & 3 \\ 3 & 2+2+7 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 3 & 2 \\ 2 & 3 & 3 \\ 3 & 2 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 3 \\ 3 & 7 & 5 \end{vmatrix}.$$

2. $$\begin{vmatrix} 1+2 & 2+2 & 5 \\ 2+3 & 1+3 & 7 \\ 4+2 & 4+2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 1 & 7 \\ 4 & 4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 7 \\ 4 & 2 & 3 \end{vmatrix}$$

$$+ \begin{vmatrix} 2 & 2 & 5 \\ 3 & 1 & 7 \\ 2 & 4 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 & 5 \\ 3 & 3 & 7 \\ 2 & 2 & 3 \end{vmatrix}.$$

Let the student verify these equations.

If, for example, all the elements of the first column are binomials, all of the second trinomials, and all of the third quadrinomials, the determinant being of the n^{th} degree, there will be $2 \times 3 \times 4$ partial determinants.

NOTE.—The development of a determinant may often be simplified by separating the elements of a column into parts, and then applying the preceding principle.

20. If all the elements of any column (or row) of a determinant be increased by equimultiples of the corresponding elements of any other column (or row), the value of the determinant will not be changed.

Let us take the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and suppose the elements of the second column to be increased by m times the corresponding elements of the third column, m being *plus* or *minus*, *entire* or *fractional*; we shall then have, from Arts. 19 and 7,

$$\begin{vmatrix} a_1 & b_1+mc_1 & c_1 \\ a_2 & b_2+mc_2 & c_2 \\ a_3 & b_3+mc_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m \begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix}.$$

But, from Art. 11, the last determinant is 0; hence,

$$\begin{vmatrix} a_1 & b_1+mc_1 & c_1 \\ a_2 & b_2+mc_2 & c_2 \\ a_3 & b_3+mc_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

In like manner, the principle may be shown to hold good for a determinant of any order.

This principle may be extended: thus, if all the elements of any column be increased by equimultiples of a second column, and also by equimultiples of a third column, and so on, the value of the determinant will not be changed.

EXAMPLE.

$$\begin{vmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{vmatrix} = \begin{vmatrix} 5+\ 6+\ 7 & 6 & 7 \\ 8+\ 9+10 & 9 & 10 \\ 11+12+13 & 12 & 13 \end{vmatrix} = 3 \begin{vmatrix} 6 & 6 & 7 \\ 9 & 9 & 10 \\ 12 & 12 & 13 \end{vmatrix} = 0.$$

Here we add the sum of the corresponding elements of the second and third columns to those of the first column; we then remove the factor 3 (Art. 7); the determinant thus found is 0, by Art. 11.

The principles just deduced are of use in simplifying the development of determinants, and still other principles might be deduced, but it is believed that those already given are sufficient for most practical purposes.

Development of Determinants.

21. The following examples, with hints for their solution, have been selected to illustrate the principles most commonly applied in the development of determinants.

EXAMPLES.

1. $$\begin{vmatrix} 4 & 2 & 7 \\ 4 & 3 & 6 \\ 4 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 9 & 7 \\ 4 & 9 & 6 \\ 4 & 9 & 5 \end{vmatrix} = 36 \begin{vmatrix} 1 & 1 & 7 \\ 1 & 1 & 6 \\ 1 & 1 & 5 \end{vmatrix} = 0.$$

HINT.—Increase the elements of the second column by those of the third (Art. 20); remove factors (Art. 7); then apply Art. 11.

2. $$\begin{vmatrix} a_1p_1+b_1q_1 & a_1p_2+b_1q_2 \\ a_2p_1+b_2q_1 & a_2p_2+b_2q_2 \end{vmatrix} = \begin{vmatrix} a_1p_1 & a_1p_2 \\ a_2p_1 & a_2p_2 \end{vmatrix} +$$

$$\begin{vmatrix} a_1p_1 & b_1q_2 \\ a_2p_1 & b_2q_2 \end{vmatrix} + \begin{vmatrix} b_1q_1 & a_1p_2 \\ b_2q_1 & a_2p_2 \end{vmatrix} + \begin{vmatrix} b_1q_1 & b_1q_2 \\ b_2q_1 & b_2q_2 \end{vmatrix}$$

$$= p_1q_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + p_2q_1 \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$$

$$= (p_1q_2 - p_2q_1) \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}.$$

HINT.—First use Art. 19; the *first* and fourth partial determinants disappear (Art. 11); the second and third partial determinants are reduced by Arts. 11 and 9; and the final result is in the form of a product of two determinants of the second degree. This example gives a formula for the multiplication of any two determinants of the second degree.

3. $$\begin{vmatrix} 4 & 3 & 4 \\ 7 & -1 & 3 \\ 2 & 8 & 6 \end{vmatrix} + \begin{vmatrix} 4 & 3 & 4 \\ 5 & 1 & -3 \\ 2 & 8 & 6 \end{vmatrix} + \begin{vmatrix} 4 & 3 & 4 \\ -12 & 0 & 0 \\ 2 & 8 & 6 \end{vmatrix} = 0.$$

HINT.—The first and third *rows* in each are identical; we therefore add the second rows element by element; all the elements of the resulting second row are 0, and consequently the determinant is 0.

4. $$\begin{vmatrix} 14 & 15 & 11 \\ 21 & 22 & 16 \\ 23 & 29 & 17 \end{vmatrix} = \begin{vmatrix} 3 & 4 & 11 \\ 5 & 6 & 16 \\ 6 & 12 & 17 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 & 11 \\ 5 & 3 & 16 \\ 6 & 6 & 17 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & 2 & 1 \\ 5 & 3 & 1 \\ 6 & 6 & -13 \end{vmatrix} = 38.$$

HINT.—Subtract the elements of the third column from those of the first and second columns (Art. 20); remove factor 2 from second column; subtract five times the elements of the second from those of the third; then reduce by Art. 2.

✓5. $$\begin{vmatrix} 1 & 1 & 1 & 4 \\ 2 & 4 & 1 & 8 \\ 4 & 1 & 2 & 13 \\ 2 & 4 & 2 & 11 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 3 & 1 & 4 \\ 2 & -1 & 2 & 5 \\ 0 & 2 & 2 & 3 \end{vmatrix} = (-1)^4 \begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 5 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= -15.$$

HINT.—Subtract third column from first and second, and four times third from fourth; then use Art. 12, *Cor.*, remembering that the place number of 1 in the upper row is $1+3=4$; finish the solution by Art. 2.

6. $$\begin{vmatrix} 5 & -10 & 11 & 0 \\ -10 & -11 & 12 & 4 \\ 11 & 12 & -11 & 2 \\ 0 & 4 & 2 & -6 \end{vmatrix} = 2\begin{vmatrix} 5 & -10 & 1 & 0 \\ -10 & -11 & 1 & 2 \\ 11 & 12 & 1 & 1 \\ 0 & 4 & 6 & -3 \end{vmatrix}$$

$$= 6\begin{vmatrix} 0 & 0 & 1 & 0 \\ -5 & -1 & 1 & 2 \\ 2 & 22 & 1 & 1 \\ -10 & 64 & 6 & -3 \end{vmatrix} = 6\begin{vmatrix} -5 & -1 & 2 \\ 2 & 22 & 1 \\ -10 & 64 & -3 \end{vmatrix} = 8100.$$

HINT.—Increase elements of third column by those of the second, and remove factor 2 from the fourth column: in result subtract five times the elements of third column from those of the first (removing factor 3), and add ten times the elements of the third column to those of the second; finish solution as in Example 5.

7. $$\begin{vmatrix} 7 & -1 & 0 & 5 \\ -2 & 3 & -2 & 2 \\ 0 & -1 & 5 & 3 \\ 5 & 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 7 & -1 & -5 & 2 \\ -2 & 3 & 13 & 11 \\ 0 & -1 & 0 & 0 \\ 5 & 1 & 8 & 7 \end{vmatrix}$$

$$= -(-1)^5\begin{vmatrix} 7 & -5 & 2 \\ -2 & 13 & 11 \\ 5 & 8 & 7 \end{vmatrix} = -486.$$

HINT.—Add five times the second to the third, and three times the second to the fourth; then multiply -1 by its sign factor $(-1)^{3+2}$, and finish the solution by Art. 2.

8. $$\begin{vmatrix} 3 & 2 & 1 & 4 \\ 6 & 3 & 3 & 2 \\ 7 & 1 & 7 & 5 \\ 5 & 7 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ -3 & -3 & 3 & -10 \\ -14 & -13 & 7 & -23 \\ -7 & -1 & 4 & -7 \end{vmatrix}$$

$$= -\begin{vmatrix} 3 & 3 & 10 \\ 14 & 13 & 23 \\ 7 & 1 & 7 \end{vmatrix} = -\begin{vmatrix} 3 & 3 & 10 \\ 0 & 11 & 9 \\ 7 & 1 & 7 \end{vmatrix} = 377.$$

9. $$\begin{vmatrix} 1 & 4 & 3 & 1 \\ 2 & 8 & 2 & 5 \\ 6 & 4 & 1 & 2 \\ 2 & 5 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -4 & 3 \\ 6 & -20 & -17 & -4 \\ 2 & -3 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 3 \\ 20 & 17 & -4 \\ 3 & 3 & 1 \end{vmatrix}.$$

$$= -101.$$

10. $$\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} \times \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = \begin{vmatrix} 8+3 & 6+6 \\ 4+5 & 3+10 \end{vmatrix} = 35.$$

Multiplication of Determinants.

22. Let us take the determinant

$$\begin{vmatrix} a_1 & b_1 & -1 & 0 \\ a_2 & b_2 & 0 & -1 \\ 0 & 0 & x_1 & y_1 \\ 0 & 0 & x_2 & y_2 \end{vmatrix} \quad \dots\dots\dots \quad (1)$$

Developing this in terms of the first column (Art 12), and then developing each subordinate determinant by Art. 12, *Cor.*, we have

$$a_1 \begin{vmatrix} b_2 & 0 & -1 \\ 0 & x_1 & y_1 \\ 0 & x_2 & y_2 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & -1 & 0 \\ 0 & x_1 & y_1 \\ 0 & x_2 & y_2 \end{vmatrix} = (a_1b_2 - a_2b_1) \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix},$$

the second member of which may be written

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}. \quad . \quad . \quad . \quad . \qquad (2)$$

Again, increase all the elements of the first column of (1) by a_1 times the corresponding elements of the third column *plus* a_2 times the corresponding elements of the fourth column; in like manner increase the elements of the second column by b_1 times those of the third *plus* b_2 times those of the fourth; we then have

$$\begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ a_1x_1+a_2y_1 & b_1x_1+b_2y_1 & x_1 & y_1 \\ a_1x_2+a_2y_2 & b_1x_2+b_2y_2 & x_2 & y_2 \end{vmatrix} \quad . \quad . \qquad (3)$$

Now, from Art. 12, *Cor.*, we may delete the first row of (3) and the third column, if we change the sign of the resulting determinant; in this result we may again delete the first row and the third column if we again change the sign of the result. Hence, (3) is equal to

$$\begin{vmatrix} a_1x_1+a_2y_1 & b_1x_1+b_2y_1 \\ a_1x_2+a_2y_2 & b_1x_2+b_2y_2 \end{vmatrix}. \quad . \quad . \quad . \qquad (4)$$

Expressions (2) and (4) are different forms of the development of (1). Hence, we have

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1x_1+a_2y_1 & b_1x_1+b_2y_1 \\ a_1x_2+a_2y_2 & b_1x_2+b_2y_2 \end{vmatrix} \quad . \qquad (5)$$

This corresponds to Ex. 2, Art. 21, and gives one form of the product of two determinants of the second order.

It will be shown hereafter that there are many forms of the product of two determinants.

From an analysis of (5), we see that one form of the product of two determinants of the second order may be found by the following

RULE.

Multiply the elements of the first row of the multiplicand by the corresponding elements of the first row of the multiplier, and take the sum of the products; then multiply the elements of the second row of the multiplicand by the corresponding elements of the first row of the multiplier and take the sum of the products: these sums constitute the first column of the product. The second column of the product is found in the same manner, using the second row of the multiplier instead of the first.

EXAMPLES.

1. $\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} \times \begin{vmatrix} 1 & 4 \\ 2 & 9 \end{vmatrix} = \begin{vmatrix} 2+12 & 4+27 \\ 3+20 & 6+45 \end{vmatrix} = \begin{vmatrix} 14 & 31 \\ 23 & 51 \end{vmatrix}.$

2. $\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} \times \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 2+2 & 3+8 \\ 6+7 & 9+28 \end{vmatrix} = \begin{vmatrix} 4 & 11 \\ 13 & 37 \end{vmatrix}.$

It has been stated that there are several forms of the product of two determinants of the second order, all of which are determinants of the second order.

It is plain that the determinant factors of the first
er of (5) can be modified either singly, or together,
verting columns into rows, and the reverse; again,

we may change the order of the two factors; and still again, we may modify both factors by an interchange of rows, or of columns, in each, or by an interchange of rows in one and of columns in the other. These modifications change the form but do not alter the value of the product. This principle may be illustrated by a special example. Thus,

$$\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} \times \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 3+8 & 1+4 \\ 9+28 & 3+14 \end{vmatrix} = \begin{vmatrix} 11 & 5 \\ 37 & 17 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \times \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 3+12 & 1+6 \\ 6+28 & 2+14 \end{vmatrix} = \begin{vmatrix} 15 & 7 \\ 34 & 16 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} \times \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = \begin{vmatrix} 3+2 & 4+4 \\ 9+7 & 12+14 \end{vmatrix} = \begin{vmatrix} 5 & 8 \\ 16 & 26 \end{vmatrix} =$$

etc., etc., etc., etc.

Let the student determine the entire number of different forms that the determinant product may take in this case.

Extension of the Rule.

23. Let us consider the following determinant:

$$\begin{vmatrix} a_1 & b_1 & c_1 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} \quad . \quad . \quad . \quad . \quad (1)$$

If we develop this determinant in terms of the first column, by Art. 12; if we then develop each subordinate determinant in terms of its first column; and if we then apply the corollary of Art. 12, we shall find that (1) may be written equal to

$$(a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1)$$

$$\times\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},$$

or to

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \quad . \quad . \quad . \quad . \quad (2)$$

Again, if we increase all of the elements of the first column of (1) by a_1 times the corresponding elements of column 4, by a_2 times the corresponding elements of column 5, and by a_3 times the corresponding elements of column 6; in like manner, if we increase all the elements of the second column by b_1, b_2, b_3 times those of the fourth, fifth, and sixth columns; and in like manner, if we increase the elements of the third column by c_1, c_2, c_3 times those of the fourth, fifth, and sixth columns, we shall have for (1),

$$\begin{vmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ a_1x_1+a_2y_1+a_3z_1 & b_1x_1+b_2y_1+b_3z_1 & c_1x_1+c_2y_1+c_3z_1 & x_1 & y_1 & z_1 \\ a_1x_2+a_2y_2+a_3z_2 & b_1x_2+b_2y_2+b_3z_2 & c_1x_2+c_2y_1+c_3z_2 & x_2 & y_2 & z_2 \\ a_1x_3+a_2y_3+a_3z_3 & b_1x_3+b_2y_3+b_3z_3 & c_1x_3+c_2y_3+c_3z_3 & x_3 & y_3 & z_3 \end{vmatrix}$$

We may reduce this result by the corollary to Art. 12, by deleting the first row and the fourth column; then, in the result, we may, for like reason, delete the first row and fourth column; then, by one more application of the same principle, we have a result which, equated with (2), gives

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$\begin{vmatrix} a_1x_1+a_2y_1+a_3z_1 & b_1x_1+b_2y_1+b_3z_1 & c_1x_1+c_2y_1+c_3z_1 \\ a_1x_2+a_2y_2+a_3z_2 & b_1x_2+b_2y_2+b_3z_2 & c_1x_2+c_2y_2+c_3z_2 \\ a_1x_3+a_2y_3+a_3z_3 & b_1x_3+b_2y_3+b_3z_3 & c_1x_3+c_2y_3+c_3z_3 \end{vmatrix} \quad . \quad . \quad . \quad (3)$$

We see from equation (3) that the rule given in the last article may be extended to find *one* form of the product of two determinants of the third order. By a similar course of reasoning, it may be further extended to finding the product of two determinants of the fourth, fifth, and higher orders.

Instead of writing out the rule in its general form, we shall illustrate its application in finding the product of two determinants of the third order by a special

EXAMPLE.

Let it be required to find *one* form of the product

$$\begin{vmatrix} 1 & 2 & 2 \\ 2 & 5 & 3 \\ 1 & 1 & 6 \end{vmatrix} \times \begin{vmatrix} 3 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 2 & 5 \end{vmatrix}.$$

Solution.—

$$\begin{vmatrix} 3+2+2 & 1+4+8 & 2+4+10 \\ 6+5+3 & 2+10+12 & 4+10+15 \\ 3+1+6 & 1+2+24 & 2+2+30 \end{vmatrix} = \begin{vmatrix} 7 & 13 & 16 \\ 14 & 24 & 29 \\ 10 & 27 & 34 \end{vmatrix}.$$

Explanation.—Multiply the elements of the first row of the multiplier by the corresponding ones in the several rows of the multiplicand, and take the sum of the results in each case; this gives the first column of the product: operate on the second and third rows of the multiplier in the same manner to get the second and third columns of the product.

ADDITIONAL EXAMPLES.

1. $$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 2 & 4 \end{vmatrix} \times \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 4+1+1 & 2+2+2 & 6+2+\ 4 \\ 2+2+2 & 1+4+4 & 3+4+\ 8 \\ 6+2+4 & 3+4+8 & 9+4+16 \end{vmatrix}.$$

2. $$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 2 & 1 & 8 \end{vmatrix} \times \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

$$= \begin{vmatrix} a+\ b+2c & d+\ e+2f & g+\ h+2k \\ 2a+2b+3c & 2d+2e+3f & 2g+2h+3k \\ 2a+\ b+8c & 2d+\ e+8f & 2g+\ h+8k \end{vmatrix}.$$

Square of a Determinant of the Third Order.

24. A simple rule for squaring a determinant of the third order may be deduced by the aid of the following principle:

The determinant $|x_1y_2z_3|$ has been shown to be equal to either of the expressions

$$x_1X_1-y_1Y_1+z_1Z_1, \quad -x_2X_2+y_2Y_2-z_2Z_2,$$
$$x_3X_3-y_3Y_3+z_3Z_3 \quad . \quad . \quad (1)$$

in which the *large* letters stand for the *principal minors* of the corresponding *small* ones.

Assuming the determinant equal to any one of these expressions, say the first, we have

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1X_1-y_1Y_1+z_1Z_1 \quad . \quad . \quad (2)$$

The quantities $X_1Y_1Z_1$ do not contain any element having the subscript $_1$; hence, if we change the subscripts of the small letters in both members of (2) from $_1$ to $_2$, these quantities will not be affected, and we shall have

$$\begin{vmatrix} x_2 & y_2 & z_2 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = x_2X_1-y_2Y_1+z_2Z_1 \quad . \quad . \quad (3)$$

The first member of (3) has two identical rows, and is therefore equal to 0 (Art. 11); hence, the second member is 0. In like manner, it may be shown that *if we change all the subscripts of the small letters in any one of expressions* (1) *to either of the other subscripts, the result will be equal to* 0.

By the rule for multiplication, we have

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} +X_1 & -Y_1 & +Z_1 \\ -X_2 & +Y_2 & -Z_2 \\ +X_3 & -Y_3 & +Z_3 \end{vmatrix} =$$

$$\begin{vmatrix} x_1X_1-y_1Y_1+z_1Z_1 & -x_1X_2+y_1Y_2-z_1Z_2 & x_1X_3-y_1Y_3+z_1Z_3 \\ x_2X_1-y_2Y_1+z_2Z_1 & -x_2X_2+y_2Y_2-z_2Z_2 & x_2X_3-y_2Y_3+z_2Z_3 \\ x_3X_1-y_3Y_1+z_3Z_1 & -x_3X_2+y_3Y_2-z_3Z_2 & x_3X_3-y_3Y_3+z_3Z_3 \end{vmatrix}.$$

But, from the principle just demonstrated, all the elements of the second member except those in the principal diagonal are equal to 0, and each of these is equal to $|x_1y_2z_3|$; hence, the second member is equal to $|x_1y_2z_3|^3$. Furthermore, the second factor of the first member is equal to $|X_1Y_2Z_3|$, as may be seen by making the development of each. Hence, we have

$$|x_1y_2z_3| \times |X_1Y_2Z_3| = |x_1y_2z_3|^3,$$

or, striking out the common factor and transposing

$$|x_1y_2z_3|^2 = |X_1Y_2Z_3|.$$

Hence, *the square of a determinant of the third order is equal to the determinant formed by replacing each of the given elements by its principal minor.*

EXAMPLE.

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 2 & 3 & 4 \end{vmatrix}^2 = \begin{vmatrix} -2 & 8 & 7 \\ 5 & 2 & -1 \\ 3 & -1 & -5 \end{vmatrix} = 121.$$

NOTE.—The elements of the second determinants are the principal minors of the corresponding elements in the first determinant.

Raising the Order of a Determinant.

25. A determinant of any order may be raised to one of the next higher order by writing 1 as a cofactor and filling the blank spaces with 0's: thus,

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & x_1 & y_1 \\ 0 & x_2 & y_2 \end{vmatrix}.$$

That the second member is equal to the first, may be shown by using Art. 12, *Cor.* It is to be noted that the 0's of the first line may be replaced by any numbers whatever: thus,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_1 & b_1 \\ 0 & a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ m_1 & a_1 & b_1 \\ m_2 & a_2 & b_2 \end{vmatrix}.$$

Multiplication of Determinants of Unequal Order.

26. If the determinant factors are of unequal order, we raise the one of lower order to the same order as the other, and proceed as in Art. 23.

EXAMPLES.

1. Find the product of $\begin{vmatrix} 14 & 15 & 11 \\ 7 & 7 & 5 \\ 2 & 7 & 1 \end{vmatrix}$ and $\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$.

Operation. $\begin{vmatrix} 14 & 15 & 11 \\ 7 & 7 & 5 \\ 2 & 7 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 3 \end{vmatrix}$

$$= \begin{vmatrix} 14 & 45+22 & 15+33 \\ 7 & 21+10 & 7+15 \\ 2 & 21+2 & 7+3 \end{vmatrix} = \begin{vmatrix} 14 & 67 & 48 \\ 7 & 31 & 22 \\ 2 & 23 & 10 \end{vmatrix}.$$

2. Find the product of $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 3 \end{vmatrix}$ and $\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 3 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 14 \\ 2 & 5 & 17 \\ 1 & 5 & 17 \end{vmatrix} = -15.$$

MISCELLANEOUS EXAMPLES.

Find the values of x in the following equations :

1. $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = 0$, or $x^3 - 3a^2x + 2a^3 = 0$.

HINT.—Add the elements of the second and third columns to those of the first column and factor ; we then have

$$(x+2a)\begin{vmatrix} 1 & a & a \\ 1 & x & a \\ 1 & a & x \end{vmatrix} = (x+2a)\,(x-a)^2 = 0.$$

Placing each factor equal to 0, we have

$$x = -2a,\ x = a,\ x = a.$$

2. $\begin{vmatrix} 2x & 5 & x \\ 3 & 3x & 2 \\ 2 & 3 & 3x \end{vmatrix} = 0$, or $18x^3 - 6x^2 - 48x + 20 = 0$.

HINT.—Proceed as in last example : we then have

$$(3x+5)\begin{vmatrix} 1 & 5 & x \\ 1 & 3x & 2 \\ 1 & 3 & 3x \end{vmatrix} = (3x+5)\,(6x^2 - 12x + 4) = 0.$$

Placing factors equal to 0, we have

$$x = -\frac{5}{3},\ x = \frac{3+\sqrt{3}}{3},\ x = \frac{3-\sqrt{3}}{3}.$$

3. $\begin{vmatrix} x & 3 & 3 \\ 3 & x & 3 \\ x^2 & 2 & 1 \end{vmatrix} = 0$, or $-3x^3 + 10x^2 - 6x + 9 = 0$.

Hint.—Subtract the elements of the second row from those of the first and factor; we then have

$$(x-3)\begin{vmatrix} 1 & -1 & 0 \\ 3 & x & 3 \\ x^2 & 2 & 1 \end{vmatrix} = -(x-3)(3x^2-x+3) = 0.$$

Ans. $x = 3,\ x = \dfrac{1 \pm \sqrt{-35}}{6}.$

Sign Factor of a Complementary Product.

27. In addition to the method of development explained in Art. 12, there are two others of great practical utility. In the *first* of these, commonly known as **Laplace's Method,** the value of the determinant is expressed in terms of complementary minors; in the *second,* which is **Cauchy's Method,** the development is made in terms of elements of one row and one column. Before explaining these methods a rule will be deduced for determining the *sign factor* that must be prefixed to the product of any two complementary minors in order that the result shall express terms of the required development.

Let us take the case of a determinant of the fourth order, which may be denoted by $\triangle$; we then have

$$\triangle = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If we delete the first two rows and the first two columns of (1), we determine two complementary minors, of which the *deleted,* or *leading* one, stands in the upper left-hand corner of the **square array** of elements; the product of these minors gives

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} = a_1b_2c_3d_4 - a_1b_2c_4d_3 - a_2b_1c_3d_4 + a_2b_1c_4d_3.$$

In this case the operation gives four terms of the development of $\triangle$, and the sign factor of the product is obviously $(-1)^0$, or 1.

Now let us delete any two rows, say the *second* and *fourth,* and any two columns, say the *second* and *third.* By 3 transfers of rows and 2 transfers of columns (Art. 8), the *deleted,* or *leading* minor, is brought to the upper left-hand corner, giving

$$-\triangle = \begin{vmatrix} b_2 & c_2 & a_2 & d_2 \\ b_4 & c_4 & a_4 & d_4 \\ b_1 & c_1 & a_1 & d_1 \\ b_3 & c_3 & a_3 & d_3 \end{vmatrix} . \quad . \quad . \quad . \quad . \qquad (2)$$

Here the product $| b_2 \ c_4 | \times | a_1 \ d_3 |$ gives four terms of the development of $-\triangle$, or with its sign changed it gives four terms of the development of $\triangle$; in this case the sign factor of the product is $(-1)^{3+2}$, or -1. Hence in any case *the sign factor of the product of two complementary minors is* $(-1)^{m+n}$ *in which* $m+n$ *is the number of transfers of rows and columns necessary to bring the leading minor to the upper left-hand corner.*

It is easily seen that this rule is applicable when the determinant is of any order; thus, in a determinant of the fifth order, $| a_1 \ b_2 \ c_3 \ d_4 \ e_5 |$, the sign factor of the complementary product, $|a_1\ c_3\ e_4| \times |b_2\ d_5|$, is $(-1)^{2+8}$, or -1.

Laplace's Method of Development.

28. Let us consider the case of a determinant of the fourth order.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} . \quad . \quad . \quad . \quad . \quad (1)$$

If we reserve any two rows, say the first two, it is plain that we can take these rows by columns so as to form six different leading minors of the second order,

$$|a_1\ b_2|,\ |a_1\ c_2|,\ |a_1\ d_2|,\ |b_1\ c_2|,\ |b_1\ d_2|,\ |c_1\ d_2|.$$

Now if we multiply each of these by its complementary minor, each product with its proper sign factor will give four terms of the development of Δ, (Art. 27), and inasmuch as these terms are all different, their aggregate will give the twenty-four terms of the required development. Hence, we have

$$\left.\begin{aligned} |a_1b_2c_3d_4| = |a_1b_2| \times |c_3d_4| - |a_1c_2| \times |b_3d_4| + |a_1d_2| \times |b_3c_4| \\ + |b_1c_2| \times |a_3d_4| - |b_1d_2| \times |a_3c_4| + |c_1d_2| \times |a_3b_4| \end{aligned}\right\} . \quad (2)$$

In like manner we develop a determinant of any order; hence, the following

RULE.

Reserve m rows, and from them form every possible minor of the mth order; multiply each of these by its *complementary minor, prefixing to each product its proper sign factor; then take the algebraic sum of the results.*

EXAMPLES.

1. Find the value of the determinant

$$\Delta = \begin{vmatrix} 8 & 7 & 2 & 20 \\ 3 & 1 & 4 & 7 \\ 5 & 0 & 11 & 0 \\ 8 & 1 & 0 & 6 \end{vmatrix}.$$

SOLUTION.—By formula (2) we have

$$\Delta = \begin{vmatrix} 8 & 7 \\ 3 & 1 \end{vmatrix} \times \begin{vmatrix} 11 & 0 \\ 0 & 6 \end{vmatrix} - \begin{vmatrix} 8 & 2 \\ 3 & 4 \end{vmatrix} \times \begin{vmatrix} 0 & 0 \\ 1 & 6 \end{vmatrix} + \begin{vmatrix} 8 & 20 \\ 3 & 7 \end{vmatrix} \times \begin{vmatrix} 0 & 11 \\ 1 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 7 & 2 \\ 1 & 4 \end{vmatrix} \times \begin{vmatrix} 5 & 0 \\ 8 & 6 \end{vmatrix} - \begin{vmatrix} 7 & 20 \\ 1 & 7 \end{vmatrix} \times \begin{vmatrix} 5 & 11 \\ 8 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 20 \\ 4 & 7 \end{vmatrix} \times \begin{vmatrix} 5 & 0 \\ 8 & 1 \end{vmatrix}$$

$$= -858 + 44 + 780 + 2552 - 330 = 2188, \text{ Ans.}$$

2. Develop the determinant

$$\Delta = \begin{vmatrix} 0 & c & b & d \\ c & 0 & a & e \\ b & a & 0 & f \\ d & e & f & 0 \end{vmatrix}.$$

SOLUTION.—Proceeding as before we have

$$\Delta = \begin{vmatrix} 0 & c \\ c & 0 \end{vmatrix} \times \begin{vmatrix} 0 & f \\ f & 0 \end{vmatrix} - \begin{vmatrix} 0 & b \\ c & a \end{vmatrix} \times \begin{vmatrix} a & f \\ e & 0 \end{vmatrix} + \begin{vmatrix} 0 & d \\ c & e \end{vmatrix} \times \begin{vmatrix} a & 0 \\ e & f \end{vmatrix}$$

$$+ \begin{vmatrix} c & b \\ 0 & a \end{vmatrix} \times \begin{vmatrix} b & f \\ d & 0 \end{vmatrix} - \begin{vmatrix} c & d \\ 0 & e \end{vmatrix} \times \begin{vmatrix} b & 0 \\ d & f \end{vmatrix} + \begin{vmatrix} b & d \\ a & e \end{vmatrix} \times \begin{vmatrix} b & a \\ d & e \end{vmatrix}$$

$$= c^2f^2 - cbef - cdaf - cdaf - cebf + (be - ad)^2$$

$$= a^2d^2 + b^3e^2 + c^2f^2 - 2\,(acdf + abde + bcef), \text{ Ans.}$$

3. Develop the determinant

$$\Delta = \begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}.$$

$$\begin{aligned} \Delta &= x^4 - x^2y^2 - x^2z^2 - x^2z^2 - x^2y^2 + (y^2 - z^2)^2 \\ &= x^4 + y^4 + z^4 - 2\,(x^2y^2 + x^2z^2 + y^2z^2), \textit{ Ans.} \end{aligned}$$

4. Develop the determinant

$$\Delta = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix}.$$

$$\Delta = x^4 + y^4 + z^4 - 2\,(x^2y^2 + x^2z^2 + y^2z^2), \textit{ Ans.}$$

Let the student apply this method of development to Examples 5 and 6, page 15.

Cauchy's Method of Development.

29. Let it be required to develop the determinant

$$\Delta = \begin{vmatrix} a_0 & x & y & z \\ x' & a_1 & b_1 & c_1 \\ y' & a_2 & b_2 & c_2 \\ z' & a_3 & b_3 & c_3 \end{vmatrix} \quad . \; . \; . \; . \; . \; . \quad (1)$$

in terms of the elements of the first column and the first row.

Let the determinant that remains after deleting the first column and the first row be denoted by Δ', that is, let

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then denote the complementary minors of the elements of Δ' by corresponding capital letters, A_1, B_1, etc. From preceding principles, all the terms in the development of Δ which contain a_0 will be found in the complementary product

$$a_0 \Delta'.$$

To find the remaining terms, we use the principle that the product of any pair of complementary minors, *with the proper sign factor,* gives terms of the development of the determinant (Art. 27).

In accordance with this principle, all the terms of Δ that contain both x' and x are found in the complementary product

$$\begin{vmatrix} a_0 & x \\ x' & a_1 \end{vmatrix} \times \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \text{ and consequently in } -x'xA_1,$$

in which A_1 is the complementary minor of that element of Δ' which lies in the same row as x' and in the same column as x.

Also, all the terms that contain both x' and y are found in the complementary product

$$-\begin{vmatrix} a_0 & y \\ x' & b_1 \end{vmatrix} \times \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \text{ or in } +x'yB_1,$$

in which B_1 is the complementary minor of that element of Δ' which corresponds to x' and y.

Also, the terms containing x' and z are found in the complementary product

$$\begin{vmatrix} a_0 & z \\ x' & c_1 \end{vmatrix} \times \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \text{ or in } -x'zC_1,$$

and so on, for the terms containing y' and x, y' and y, etc.

Then, because each term of the development of Δ must contain one element, and only one, from the first column, and one element, and only one, from the first row, we have

$$\left.\begin{aligned} \Delta = a_0\Delta' - x'xA_1 + x'yB_1 - x'zC_1 \\ + y'xA_2 - y'yB_2 + y'zC_2 \\ - z'xA_3 + z'yB_3 - z'zC_3 \end{aligned}\right\} . \quad . \quad (3)$$

Formula (3) enables us to develop a determinant of the fourth order in terms of the elements of its first column and first row. In like manner we may deduce a formula for developing a determinant of any order.

A determinant can be developed in terms of any column and any row by transferring the column to the left-hand *border* and the row to the upper *border*, due regard being paid to the sign of the resulting determinant.

In the case we have considered, the determinant Δ' is said to be **bordered** by the upper row and the left-hand column.

Bordered determinants are of frequent occurrence in analysis.

EXAMPLES.

1. Develop by the preceding formula,

$$\Delta = \begin{vmatrix} 0 & x & y & z \\ x & a & h & g \\ y & h & b & f \\ z & g & f & c \end{vmatrix}.$$

SOLUTION. $\Delta = -x^2(bc-f^2)+xy(hc-gf)-xz(hf-gb)$
$+yx(hc-fg)-y^2(ac-g^2)+yz(af-gh)$
$-zx(hf-bg)+zy(af-hg)-z^2(ab-h^2)$.

Whence, by reduction,

$$\Delta = x^2(f^2-bc)+y^2(g^2-ac)+z^2(h^2-ab)$$
$$+2[xy(hc-gf)+xz(bg-hf)+yz(af-gh)], \textit{ Ans.}$$

2. Develop the numerical determinant

$$\Delta = \begin{vmatrix} 0 & 1 & -13 & 6 \\ 4 & 3 & 9 & 2 \\ 8 & 9 & 6 & 12 \\ 3 & 1 & 2 & 4 \end{vmatrix}.$$

SOLUTION.

$$\Delta = \left.\begin{array}{l} -4\times 0-\ 52\times 24-24\times\ 12 \\ +8\times 32+104\times 10+48\times -3 \\ -3\times 96-\ 39\times 18-18\times -63 \end{array}\right\} = -240, \textit{ Ans.}$$

Let the student apply the foregoing formula to the solution of the examples given in the preceding article.

Symmetrical Determinants.

30. Two elements of a determinant are said to be **Conjugate** when the row and column numbers of one, are the column and row numbers of the other. Thus, the element standing in the second row and third column of a determinant is the conjugate of the element standing in the second column and third row. Each element of the principal diagonal is its own conjugate.

A **Symmetrical** determinant is one in which each element is equal to its conjugate; thus, the determinant

$$\Delta = \begin{vmatrix} a_0 & x & y & z \\ x & a_1 & b_1 & c_1 \\ y & b_1 & b_2 & c_2 \\ z & c_1 & c_2 & c_3 \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is a symmetrical determinant, the principal diagonal being the **Axis of Symmetry**.

If all the elements of the axis are 0, the determinant is said to be **Zero-axial**. Thus, the determinants in Examples 2 and 3 of Art. 28 are *zero-axial*.

The form of a symmetrical determinant is not altered by changing rows to columns, and the reverse; hence, the complementary minor of any element is equal to that of its conjugate.

The **Principal Minors** of a determinant are those whose diagonals coincide with the principal diagonal of the determinant; it is easily seen that the principal minors of a symmetrical determinant are themselves symmetrical.

Symmetrical determinants, which frequently occur in analysis, are usually developed by the method of Art. 29. Applying this method to determinant (1) and reducing we have

$$\Delta = a_0 \Delta' - (x^2 A_1 + y^2 B_2 + z^2 C_3) \\ + 2\,[xyB_1 - xzC_1 + yzC_2]. \quad . \quad (2)$$

This result comes at once from formula (3), Art. 29, by making the border symmetrical, and then making $A_2 = B_1$, $A_3 = C_1$, and $B_3 = C_2$.

EXAMPLES.

1. Develop the equation

$$\begin{vmatrix} 0 & x & y & z \\ x & 1 & 2 & 3 \\ y & 2 & 1 & 4 \\ z & 3 & 4 & 1 \end{vmatrix} = 0.$$

HINT.—Apply formula (2) and we have

$$15x^2 + 8y^2 + 3z^2 - 20xy - 10xz - 4yz = 0, \textit{ Ans.}$$

2. Develop the determinant

$$\Delta = - \begin{vmatrix} 0 & bc & ca & ab \\ bc & 1 & \cos\alpha & \cos\beta \\ ca & \cos\alpha & 1 & \cos\gamma \\ ab & \cos\beta & \cos\gamma & 1 \end{vmatrix}.$$

HINT.—By formula (2) we have

$$\Delta = b^2c^2(1-\cos^2\gamma) + c^2a^2(1-\cos^2\beta) + a^2b^2(1-\cos^2\alpha) \\ -2\left\{\begin{array}{l} bc \cdot ca\ (\cos\alpha - \cos\beta\cos\gamma \\ bc \cdot ab\ (\cos\beta - \cos\gamma\cos\alpha \\ ca \cdot ab\ (\cos\gamma - \cos\alpha\cos\beta \end{array}\right\}, \textit{ Ans.}$$

3. Find the value of the determinant

$$\Delta = \begin{vmatrix} k & h & g & x \\ h & j & f & y \\ g & f & c & z \\ x & y & z & 0 \end{vmatrix}.$$

HINT.—We first transfer the 4th column to the first place, and then transfer the 4th row to the first place, giving

$$\Delta = \begin{vmatrix} 0 & x & y & z \\ x & k & h & g \\ y & h & j & f \\ z & g & f & c \end{vmatrix}.$$

Then by applying formula (2), we have

$$\begin{aligned} \Delta &= x^2(f^2-jc)+y^2(g^2-kc)+z^2(h^2-kj) \\ &\quad +2\,[xy\,(hc-gf)+xz\,(gj-hf)+yz\,(kf-gh)], \ \textit{Ans.} \end{aligned}$$

It is to be noted that the capital letters in formula (2) stand for the complementary minors of the corresponding small letters in Δ'.

Skew Determinants.

31. A **Skew Determinant** is one in which every element on one side of the principal diagonal is equal to its conjugate with its sign changed; thus,

$$\Delta = \begin{vmatrix} \alpha & b_1 & c_1 & d_1 \\ -b_1 & \beta & c_2 & d_2 \\ -c_1 & -c_2 & \gamma & d_3 \\ -d_1 & -d_2 & -d_3 & \delta \end{vmatrix} \quad . \quad . \quad . \quad (1)$$

is a skew determinant.

A Skew Symmetrical Determinant is one in which every element is equal to its conjugate with its sign changed.

Because every element of the principal diagonal is its own conjugate, and because 0 is the only expression whose value does not change with a change of sign, it follows that all the elements of the principal diagonal are 0; that is, the determinant is *zero-axial.* Hence,

$$\triangle = \begin{vmatrix} 0 & x & y & z \\ -x & 0 & b_1 & c_1 \\ -y & -b_1 & 0 & c_2 \\ -z & -c_1 & -c_2 & 0 \end{vmatrix} \quad . \quad . \quad . \quad . \quad (1)$$

is a *skew symmetrical* determinant.

In treating determinants of this kind, attention should be given to the following principles:

1°. If the *columns* of a skew symmetrical are made *rows,* and then if the signs of the elements in each row are changed, the **array** of elements in the result will be the same as in the original determinant, but the value of $\triangle$ will become $(-1)^n \triangle$, (Art. 7), in which n denotes the order of the determinant. Now if n is *odd,* the array will remain the same, but the sign of $\triangle$ will be changed; hence, $\triangle = 0$, that is, a skew symmetrical determinant of an odd order is equal to 0.

2°. The complementary minor of any element may be found from that of its conjugate by changing the columns of the latter to rows and then changing the signs of the elements of every row. Now, because the *order* of the complementary minor of an element is less by 1 than that of the given determinant, it follows that the complementary minors of two conjugate elements are *equal* when the determinant is of an odd order, and *equal with contrary signs* when the determinant is of an even order. Thus, in (1) the complementary minors of c_2 and $-c_2$ are equal with contrary signs, as may easily be verified.

As an illustration of the method of developing a *skew symmetrical determinant,* let us apply Cauchy's method to the example just given, using for that purpose formula (3), Art. 29.

By comparing determinant (1), in Art. 29, with determinant (1) in the present article, we see that $a_0 = 0$; x', y', z' are replaced by $-x$, $-y$, $-z$; $A_1 = c_2{}^2$; $B_1 = A_2 = c_1c_2$; $C_1 = A_3 = b_1c_2$; $B_2 = c_1{}^2$; $C_2 = B_3 = b_1c_1$; $C_3 = b_1{}^2$. Making these changes and arranging terms, we have

$$\begin{aligned}\Delta &= c_2{}^2x^2 - 2c_1c_2xy + c_1{}^2y^2 + 2b_1z(c_2x - c_1y) + b_1{}^2z^2 \\ &= (c_2x - c_1y + b_1z)^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)\end{aligned}$$

It may be shown that every skew symmetrical determinant of an even order is a perfect square.

EXAMPLES.

1. Develop the determinant

$$\Delta = \begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}.$$

HINT.—Apply formula (2) and we have

$$\triangle = (af-be+cd)^2, \quad \textit{Ans.}$$

2. Develop the determinant

$$\triangle = \begin{vmatrix} 0 & 2 & 1 & 3 \\ -2 & 0 & 4 & 2 \\ -1 & -4 & 0 & 5 \\ -3 & -2 & -5 & 0 \end{vmatrix}.$$

$$\textit{Ans.} \quad \triangle = (10-2+12)^2 = 400.$$

Reciprocal Determinants.

32. If each element of a determinant is replaced by its *co-factor,* the resulting determinant is called the **Reciprocal** of the given determinant; thus, of the determinants

$$\triangle = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \text{ and } \triangle' = \begin{vmatrix} X_1 & -Y_1 & Z_1 \\ -X_2 & Y_2 & -Z_2 \\ X_3 & -Y_3 & Z_3 \end{vmatrix},$$

$\triangle'$ is the reciprocal determinant of $\triangle$. The capital letters in $\triangle'$ stand for the complementary minors of the corresponding small letters in $\triangle$, their signs being + when the sums of their row and column numbers is *even* and — when these sums are *odd.* Reciprocal determinants are sometimes called **Adjugate Determinants.**

It is shown in Art. 24 that the reciprocal of a determinant of the third order is equal to the square of that determinant. It may be shown that the reciprocal of a determinant of the n^{th} order is equal to the $(n-1)$ power of that determinant.

EXAMPLES.

1. Write out the reciprocal determinant of

$$\Delta = \begin{vmatrix} k & h & g \\ h & j & f \\ g & f & c \end{vmatrix},$$

in terms of the elements.

$$\Delta' = \begin{vmatrix} cj-f_2 & fg-hc & hf-gj \\ fg-hc & kc-g^2 & gh-fk \\ hf-gj & gh-fk & kj-h^2 \end{vmatrix}, \textit{ Ans.}$$

We see that Δ' is a symmetrical determinant.

2. Express the reciprocal of the following determinant in terms of its elements.

$$\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}.$$

$$\Delta' = \begin{vmatrix} c^2 & -bc & ac \\ -bc & b^2 & -ab \\ ac & -ab & a^2 \end{vmatrix}, \textit{ Ans.}$$

Both Δ and Δ' are equal to 0.

Rectangular Arrays.

33. In the written expression of a determinant the elements are arranged in what is called a **Square Array.** If we delete one or more columns, or if we delete one or more rows, the remaining elements form what is called a **Rectangular Array.** A rectangular array may have more rows than columns, or it may contain more columns than rows. Thus, (1) is an array of the former kind and (2) is one of the latter kind.

$$\left.\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}\right\} \ldots \quad (1) \qquad \left.\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right\} \ldots \quad (2)$$

Rectangular arrays by themselves do not represent functions, but if two such arrays of the same dimensions are compounded by the method of multiplication (Art. 23), they give rise to determinants which are of importance in analysis.

1°. Let us apply the rule of Art. 23 to find what may be called the *product* of two arrays of the first kind, as

$$\left.\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{matrix}\right\}.$$

Denoting the result of the operation by $\triangle$, we have

$$\triangle = \begin{vmatrix} a_1x_1+b_1y_1 & a_1x_2+b_1y_2 & a_1x_3+b_1y_3 \\ a_2x_1+b_2y_1 & a_2x_2+b_2y_2 & a_2x_3+b_2y_3 \\ a_3x_1+b_3y_1 & a_3x_2+b_3y_2 & a_3x_3+b_3y_3 \end{vmatrix}. \quad (1)$$

The value of $\triangle$ is plainly equal to 0, for it is the same as the product of the two determinants that would result by annexing a column of 0's to each of the given arrays.

This principle may be extended to the multiplication of any two arrays of equal dimensions and of the first kind. In all cases the resulting determinant is equal to 0.

2°. Let us apply the rule of Art. 23 to find the product of two arrays of the second kind,

$$\left.\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{matrix}\right\}.$$

Denoting the result by $\triangle$, we have

$$\triangle = \begin{vmatrix} a_1x_1+b_1y_1+c_1z_1 & a_1x_2+b_1y_2+c_1z_2 \\ a_2x_1+b_2y_1+c_2z_1 & a_2x_2+b_2y_2+c_2z_2 \end{vmatrix}. \qquad (2)$$

Now, if we find the development of (2) by the ordinary method it will contain 18 terms, of which 9 are *plus* and 9 are *minus*. The terms containing x_1x_2, y_1y_2, z_1z_2, will obviously neutralize each other. Those containing x_1y_2 give $x_1y_2\ (a_1b_2-a_2b_1)$; and those containing x_2y_1 give $x_2y_1\ (a_2b_1-a_1b_2)$; the aggregate of these is $(a_1b_2-a_2b_1)(x_1y_2-x_2y_1)$. In like manner the aggregate of the terms containing x_1z_2 and x_2z_1 give $(a_1c_2-a_2c_1)(x_1z_2-x_2z_1)$. And similarly for the terms containing y_1z_2 and y_2z_1, we have the expression $(b_1c_2-b_2c_1)(y_1z_2-y_2z_1)$. We then have

$$\triangle = \begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix} \times \begin{vmatrix} x_1y_1 \\ x_2y_2 \end{vmatrix} + \begin{vmatrix} a_1c_1 \\ a_2c_2 \end{vmatrix} \times \begin{vmatrix} x_1z_1 \\ x_2z_2 \end{vmatrix} + \begin{vmatrix} b_1c_1 \\ b_2c_2 \end{vmatrix} \times \begin{vmatrix} y_1z_1 \\ y_2z_2 \end{vmatrix}. \quad . \qquad (3)$$

Hence, in this case, *the determinant formed by multiplying two arrays of equal dimensions and of the second kind, is equal to the sum of all the determinants that be found from one of the arrays taken by columns, each multiplied by the corresponding determinant formed from the other array.*

The above principle may be extended to any case, in which the arrays are of the same dimensions and of the second kind.

EXAMPLES.

1. From the two arrays

$$\left.\begin{matrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{matrix}\right\} \text{ and } \left.\begin{matrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{matrix}\right\}, \text{ show that}$$

$$\begin{vmatrix} 3 & \alpha+\beta+\gamma \\ \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 \end{vmatrix} = (\alpha-\beta)^2+(\alpha-\gamma)^2+(\beta-\gamma)^2.$$

Multiplying the arrays and applying the principle above demonstrated, we have

$$\begin{vmatrix} 1+1+1 & \alpha+\beta+\gamma \\ \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \alpha & \beta \end{vmatrix} \times \begin{vmatrix} 1 & 1 \\ \alpha & \beta \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ \alpha & \gamma \end{vmatrix} \times \begin{vmatrix} 1 & 1 \\ \alpha & \gamma \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 1 \\ \beta & \gamma \end{vmatrix} \times \begin{vmatrix} 1 & 1 \\ \beta & \gamma \end{vmatrix} = (\alpha-\beta)^2+(\alpha-\gamma)^2+(\beta-\gamma)^2.$$

2. From the two arrays

$$\left.\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right\} \text{ and } \left.\begin{matrix} c_1 & -2b_1 & a_1 \\ c_2 & -2b_2 & a_2 \end{matrix}\right\},$$

find the value of $4\,(a_1c_1-b_1{}^2) \times (a_2c_2-b_2{}^2)$.

Solution.—Multiplying the two arrays and reducing, we have

$$\begin{vmatrix} 2\,(a_1c_1-b_1{}^2) & a_1c_2-2b_1b_2+a_2c_1 \\ a_2c_1-2b_1b_2+a_1c_2 & 2\,(a_2c_2-b_2{}^2) \end{vmatrix}. \quad . \quad (4)$$

By the principle demonstrated, (4) is equal to

$$-2\begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix} \times \begin{vmatrix} c_1b_1 \\ c_2b_2 \end{vmatrix} + \begin{vmatrix} a_1c_1 \\ a_2c_2 \end{vmatrix} \times \begin{vmatrix} c_1a_1 \\ c_2a_2 \end{vmatrix} - 2\begin{vmatrix} b_1c_1 \\ b_2c_2 \end{vmatrix} \times \begin{vmatrix} b_1a_1 \\ b_2a_2 \end{vmatrix}. \quad . \quad (5)$$

The determinant (4) gives, on reduction,

$$4\,(a_1c_1-b_1{}^2)\,(a_2c_2-b_2{}^2)-(a_1c_2-2b_1b_2+a_2c_1)^2. \quad (6)$$

The first and third of determinants (5) are equal and their aggregate gives $4\,(a_1b_2-a_2b_1)\,(b_1c_2-b_2c_1)$; and the second determinant of (5) gives $-(a_1c_2-a_2c_1)^2$.

Equating and transposing, we have

$$\begin{aligned}4\,(a_1c_1-b_1{}^2)\,(a_2c_2-b_2{}^2) &= (a_1c_2-2b_1b_2+a_2c_1)^2\\ &\quad+4\,(a_1b_2-a_2b_1)\,(b_1c_2-b_2c_1)-(a_1c_2-a_2c_1)^2,\ \textit{Ans.}\end{aligned}$$

Rectangular Arrays in Notation.

34. From the rectangular array

$$\left.\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}\right\}$$

We can form three determinants whose columns are the columns of the given array,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.$$

To express the fact that each of these is equal to 0, we draw double bars before and after the array and then place the result equal to 0; thus,

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

As an application of this method of notation let us take the group of simultaneous equations,

$$a_1x+b_1y=0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$a_2x+b_2y=0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$a_3x+b_3y=0. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If these equations are simultaneous, they must be simultaneous in pairs. If (2) and (3) consist, we have from Art. 16 the condition

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0.$$

If (2) and (4) consist, we have

$$\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = 0.$$

If (3) and (4) consist, we have

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0.$$

Hence, if the entire group of equations are consistent, we have

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Vmatrix} = 0.$$

In like manner if the group

$$\begin{aligned} a_1x+b_1y+c_1z &= 0 \\ a_2x+b_2y+c_2z &= 0 \\ a_3x+b_3y+c_3z &= 0 \\ a_4x+b_4y+c_4z &= 0 \end{aligned}$$

are consistent, we have

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{Vmatrix} = 0; \quad . \quad . \quad . \quad . \quad (5)$$

that is, each of the determinants of the third order that can be formed by taking the columns of (5), is separately equal to 0.

Other Modes of Notation.

35. It was shown that a determinant, say of the third degree, may be written in either of the forms,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad | a_1 \; b_2 \; c_3 |, \quad \text{or} \quad \Sigma \pm a_1 b_2 c_3.$$

A modification of this mode of notation consists in representing elements by a single letter with double subscript numbers, the first of which indicates the *row* and the second the *column* in which the element is found; thus, the general determinant of the fourth order may be written

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It is to be observed that a_{32} and a_{23} are different symbols, the former denoting the element in the third row and second column, and the latter the element in the second row and third column; in fact, the two symbols stand for conjugate elements.

The determinant (1) may be written

$$| a_{11} \; a_{22} \; a_{33} \; a_{44} |, \quad \text{or} \quad \Sigma \pm a_{11} \; a_{22} \; a_{33} \; a_{44}.$$

The general term of a determinant is denoted by a_{ks}, in which k and s may have any values from 1 to n inclusive. The determinant may then be written

$$| a_{ks} |, \quad (k, s = 1 \cdot 2 \; . \; . \; . \; n).$$

In many cases the bracket can be omitted.

Differential of a Determinant.

36. Let us first consider a determinant of the second order. We have

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1 ; \quad . \quad . \quad . \quad . \quad (1)$$

hence,

$$d\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = y_2dx_1 + x_1dy_2 - y_1dx_2 - x_2dy_1 \quad . \quad (2)$$

which may be written

$$d\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} dx_1 & y_1 \\ dx_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_1 & dy_1 \\ x_2 & dy_2 \end{vmatrix}.$$

Hence, the differential of a determinant of the second order is equal to the sum of two determinants formed from the original determinant, the *first* by replacing the elements of the first column by their differentials, and the *second* by replacing the elements of the second column by their differentials.

Extension of the Principle.

37. Next, let us consider a determinant of the third order, which, when developed by the method of Art. 12, gives

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1X_1 - x_2X_2 + x_3X_3 \quad . \quad . \quad (1)$$

Differentiating the second member of (1), we have

$$X_1dx_1 - X_2dx_2 + X_3dx_3 + x_1dX_1 - x_2dX_2 + x_3dX_3 \quad . \quad (2)$$

Differentiating X_1, X_2, and X_3 by the method of the last article, remembering that

$$X_1 = \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix}, \quad X_2 = \begin{vmatrix} y_1 & z_1 \\ y_3 & z_3 \end{vmatrix}, \text{ and } X_3 = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix},$$

and multiplying by x_1, x_2, x_3, we have for the last three terms of expression (2),

$$\left.\begin{aligned} x_1 dX_1 &= x_1 \begin{vmatrix} dy_2 & z_2 \\ dy_3 & z_3 \end{vmatrix} + x_1 \begin{vmatrix} y_2 & dz_2 \\ y_3 & dz_3 \end{vmatrix} \\ -x_2 dX_2 &= -x_2 \begin{vmatrix} dy_1 & z_1 \\ dy_3 & z_3 \end{vmatrix} - x_2 \begin{vmatrix} y_1 & dz_1 \\ y_3 & dz_3 \end{vmatrix} \\ x_3 dX_3 &= x_3 \begin{vmatrix} dy_1 & z_1 \\ dy_2 & z_2 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & dz_1 \\ y_2 & dz_2 \end{vmatrix} \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

The sum of the first three terms in expression (2) is a determinant, the sum of the first column of the second members of equations (3) is also a determinant, and so is the sum of the second column of the second members of (3). Writing them in determinant form, we have

$$d\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} dx_1 & y_1 & z_1 \\ dx_2 & y_2 & z_2 \\ dx_3 & y_3 & z_3 \end{vmatrix} + \begin{vmatrix} x_1 & dy_1 & z_1 \\ x_2 & dy_2 & z_2 \\ x_3 & dy_3 & z_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & dz_1 \\ x_2 & y_2 & dz_2 \\ x_3 & y_3 & dz_3 \end{vmatrix}.$$

In this case the differential is equal to the sum of three determinants formed from the original determinant by replacing in succession the elements of the respective columns by their differentials.

By a similar course of reasoning, we may extend the principle to determinants of any order.

EXAMPLES.

1. $$d\begin{vmatrix} 2x & 5 & x \\ 3 & 3x & 2 \\ 2 & 3 & 3x \end{vmatrix} = \begin{vmatrix} 2dx & 5 & x \\ 0 & 3x & 2 \\ 0 & 3 & 3x \end{vmatrix} + \begin{vmatrix} 2x & 0 & x \\ 3 & 3dx & 2 \\ 2 & 0 & 3x \end{vmatrix} + \begin{vmatrix} 2x & 5 & dx \\ 3 & 3x & 0 \\ 2 & 3 & 3dx \end{vmatrix}$$

$$= dx\,(18x^2-12+18x^2-6x+18x^2+9-6x-45)$$
$$= (54x^2-12x-48)\,dx.$$

2. $$d\begin{vmatrix} x & 3 & 3 \\ 3 & x & 3 \\ x^2 & 2 & 1 \end{vmatrix} = \begin{vmatrix} dx & 3 & 3 \\ 0 & x & 3 \\ 2xdx & 2 & 1 \end{vmatrix} + \begin{vmatrix} x & 0 & 3 \\ 3 & dx & 3 \\ x^2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} x & 3 & 0 \\ 3 & x & 0 \\ x^2 & 2 & 0 \end{vmatrix}$$

$$= dx\,(x+18x-6x^2-6+x-3x^2)$$
$$= (-9x^2+20x-6)\,dx.$$

MISCELLANEOUS EXAMPLES.

1. Develop the determinant

$$\triangle = \begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix}. \quad . \quad . \quad . \quad (1)$$

Hint.—By Laplace's method, after reduction,

$$\triangle = (a^2+b^2)^2+2(ad-bc)^2+2(ac+bd)^2+(c^2+d^2)^2, \text{ or}$$

$$\triangle = (a^2+b^2+c^2+d^2)^2, \textit{ Ans.}$$

2. Develop the determinant

$$\triangle' = \begin{vmatrix} p & q & r & s \\ -q & p & -s & r \\ -r & s & p & -q \\ -s & -r & q & p \end{vmatrix}. \quad . \quad . \quad . \quad (2)$$

Hint.—Substitute p for a, q for b, etc., in Ex. 1, and we have

$$\Delta' = (p^2+q^2+r^2+s^2)^2, \textit{ Ans.}$$

3. Find the product of (1) and (2) in the preceding examples, after making

$$ap+bq+cr+ds = A\,; \quad -aq+bp-cs+dr = B\,;$$

$$-ar+bs+cp-dq = C\,; \quad -as-br+cq+dp = D.$$

Calling the product Δ'', we have

$$\Delta'' = \begin{vmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{vmatrix} = (A^2+B^2+C^2+D^2)^2, \textit{ Ans.}$$

Comparing examples 1, 2, and 3, we see that

$$\sqrt{\Delta\,\Delta'} = \sqrt{\Delta''}, \quad \text{or}$$

that the product of two quantities, each the sum of four squares, is itself the sum of four squares. This is called Euler's theorem.

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